

CANONICAL CONNECTIONS ON PARACONTACT MANIFOLDS

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ABSTRACT. The canonical paracontact connection is defined and it is shown that its torsion is the obstruction the paracontact manifold to be paraSasakian. A \mathcal{D} -homothetic transformation is determined as a special gauge transformation. The η -Einstein manifold are defined, it is prove that their scalar curvature is a constant and it is shown that in the paraSasakian case these spaces can be obtained from Einstein paraSasakian manifolds with a \mathcal{D} -homothetic transformations. It is shown that an almost paracontact structure admits a connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor of the paracontact structure is skew-symmetric and the defining vector field is Killing.

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1. INTRODUCTION

In [6] Kaneyuki and Konzai defined the almost paracontact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ and constructed the almost paracomplex structure on $M^{(2n+1)} \times \mathbb{R}$. In this paper we study the properties of an almost paracontact metric manifold. We consider gauge (conformal) transformations of a paracontact manifold i.e. transformations preserving the paracontact structure. We define \mathcal{D} -homothetic transformations as a special gauge transformation (homothetic) and study the behavior of the Einstein condition under \mathcal{D} -homothetic transformations on a paracontact metric manifold. We consider the η -Einstein manifold, prove that their scalar curvature is a constant and show that in the paraSasakian case these spaces are the images of Einstein paraSasakian manifolds under \mathcal{D} -homothetic transformations.

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We define a canonical paracontact connection on a paracontact metric manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection. We show that the torsion of this connection vanishes exactly when the structure is para-Sasakian and compute the gauge transformation of its scalar curvature.

We introduce and study also the notion of paracontact manifolds with torsion. The paracontact manifolds with torsion are manifolds, which admit a linear almost paracontact connection with totally skew-symmetric torsion. We prove that an almost paracontact structure admits a connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor of the paracontact structure is skew-symmetric and the defining vector field is Killing. In the contact case this connection is studied in [3].

2. ALMOST PARACONTACT MANIFOLDS

A $(2n+1)$ -dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

- (i) $\varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$
 - (ii) $\eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi,$
- (2.1) (iii) let $\mathbb{D} = Ker \eta$ be the horizontal distribution generated by η , then the tensor field φ induces an almost paracomplex structure (see [5]) on each fibre on \mathbb{D} .

Recall that an almost paracomplex structure on an $2n$ -dimensional manifold is a $(1,1)$ -tensor J such that $J^2 = 1$ and the eigensubbundles T^+, T^- corresponding to the eigenvalues $1, -1$ of J , respectively have equal dimension n . The Nijenhuis tensor N of J , given by $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$, is the obstruction for the integrability of the eigensubbundles T^+, T^- . If $N = 0$ then the almost paracomplex structure is called paracomplex or integrable.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism φ has rank $2n$, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, (see [1, 2] for the almost contact case).

If a manifold $M^{(2n+1)}$ with (φ, ξ, η) -structure admits a pseudo-Riemannian metric g such that

$$(2.2) \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible* metric. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$.

Any almost paracontact structure admits a compatible metric. Indeed, if G is any metric, first set $\overline{G}(X, Y) = G(\varphi^2 X, \varphi^2 Y) + \eta(X)\eta(Y)$; then $\eta(X) = \overline{G}(X, \xi)$. Now define g by $g(X, Y) = \frac{1}{2}(\overline{G}(X, Y) - \overline{G}(\varphi X, \varphi Y) + \eta(X)\eta(Y))$ and check g is compatible.

The fundamental 2-form

$$(2.3) \quad F(X, Y) = g(X, \varphi Y)$$

is non-degenerate on the horizontal distribution \mathbb{D} and $\eta \wedge F^n \neq 0$.

Definition 2.1. If $g(X, \varphi Y) = d\eta(X, Y)$ (where $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]))$) then η is a paracontact form and the almost paracontact metric manifold (M, φ, η, g) is said to be *paracontact metric manifold*.

The manifold M is orientable exactly when the canonical line bundle $E = \{\eta \in \Lambda^1 : \text{Ker } \eta = \mathbb{D}\}$ is orientable, since \mathbb{D} is orientable by the paracomplex structure φ . Any two contact forms $\bar{\eta}, \eta \in E$ are connected by

$$(2.4) \quad \bar{\eta} = \sigma\eta,$$

where σ is non-vanishing smooth function on M . We study this conformal (gauge) transformation in *Section 4*

Remark 2.2. We mention that some authors say $M^{(2n+1)}$ has an almost paracontact metric structure if it admits a Riemannian metric g such that $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ (see [8, 4]). In our paper the metric is a pseudo-Riemannian and metric satisfies a condition (2.2)

For a manifold $M^{(2n+1)}$ with an almost paracontact metric structure (φ, ξ, η, g) we can also construct a useful local orthonormal basis. Let U be a coordinate neighborhood on M and X_1 any unit vector field on U orthogonal to ξ . Then φX_1 is a vector field orthogonal to both X and ξ , and $|\varphi X_1|^2 = -1$. Now choose a unit vector field X_2 orthogonal to ξ , X_1 and φX_1 . Then φX_2 is also vector field orthogonal to ξ , X_1 , φX_1 and X_2 , and $|\varphi X_2|^2 = -1$. Proceeding in this way we obtain a local orthonormal basis $(X_i, \varphi X_i, \xi), i = 1 \dots n$ called a φ -basis.

Hence, an almost paracontact metric manifold $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ is an odd dimensional manifold with a structure group $\mathbb{U}(n, \mathbb{R}) \times \text{Id}$, where $\mathbb{U}(n, \mathbb{R})$ is the para-unitary group isomorphic to $\mathbb{GL}(n, \mathbb{R})$.

Let $M^{(2n+1)}$ be an almost paracontact manifold with structure (φ, ξ, η) and consider the manifold $M^{(2n+1)} \times \mathbb{R}$. We denote a vector field on $M^{(2n+1)} \times \mathbb{R}$ by $(X, f \frac{d}{dt})$ where X is tangent to $M^{(2n+1)}$, t is the coordinate on \mathbb{R} and f is a C^∞ function on $M^{(2n+1)} \times \mathbb{R}$. An almost paracomplex structure J on $M^{(2n+1)} \times \mathbb{R}$ is defined in [6] by

$$J(X, f \frac{d}{dt}) = (\varphi X + f\xi, \eta(X) \frac{d}{dt}).$$

If J is integrable, we say that the almost paracontact structure (φ, ξ, η) is *normal*.

As the vanishing of the Nijenhuis tensor of J is necessary and sufficient condition for integrability, we express the condition of normality in terms of Nijenhuis tensor of φ . Since N_J is tensor field of type $(1, 2)$, it suffices to compute $N_J((X, 0), (Y, 0))$ and $N_J((X, 0), (0, \frac{d}{dt}))$ for vector fields X and Y on $M^{(2n+1)}$.

$$N_J((X, 0), (Y, 0)) = ([X, Y], 0) + ([\varphi X, \varphi Y], (\varphi X\eta(Y) - \varphi Y\eta(X)) \frac{d}{dt}) -$$

$$\begin{aligned}
& -(\varphi[\varphi X, Y] - Y\eta(X)\xi, \eta([\varphi X, Y])\frac{d}{dt}) - (\varphi[X, \varphi Y] + X\eta(Y)\xi, \eta([X, \varphi Y])\frac{d}{dt}) = \\
& (N_\varphi(X, Y) - 2d\eta(X, Y)\xi, ((\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X)\frac{d}{dt}). \\
& N_J((X, 0), (0, \frac{d}{dt})) = [(\varphi X, \eta(X)\frac{d}{dt}), (\xi, 0)] - J[(X, 0), (\xi, 0)] = \\
& = ([\varphi X, \xi], -\xi(\eta(X))\frac{d}{dt}) - (\varphi[X, \xi], \eta([X, \xi])\frac{d}{dt}) = \\
& = -((\mathcal{L}_\xi\varphi)X, (\mathcal{L}_\xi\eta)X\frac{d}{dt}).
\end{aligned}$$

We are thus lead to define tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ by

$$\begin{aligned}
N^{(1)}(X, Y) &= N_\varphi(X, Y) - 2d\eta(X, Y)\xi, \\
N^{(2)}(X, Y) &= (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \\
N^{(3)}(X) &= (\mathcal{L}_\xi\varphi)X, \\
N^{(4)}(X) &= (\mathcal{L}_\xi\eta)X.
\end{aligned}$$

Clearly the almost paracontact structure (φ, ξ, η) is normal if and only if these four tensors vanish.

Proposition 2.3. *For an almost paracontact structure (φ, ξ, η) the vanishing of $N^{(1)}$ implies the vanishing $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$;*

For a paracontact structure (φ, ξ, η, g) , $N^{(2)}$ and $N^{(4)}$ vanish. Moreover $N^{(3)}$ vanishes if and only if ξ is a Killing vector field.

Proof. Setting $Y = \xi$ in $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]))$ and we get $d\eta(X, \xi) = 0$. We have

$$0 = N_\varphi(X, \xi) = -\varphi[\varphi X, \xi] + \varphi^2[X, \xi] = \varphi((\mathcal{L}_\xi\varphi)X).$$

Applying φ and noting that $d\eta(\varphi X, \xi) = 0$ implies $\eta([\xi, \varphi X]) = 0$, we have $N^{(3)} = 0$. Moreover $(\mathcal{L}_\xi\eta)\varphi X = 0$, but $(\mathcal{L}_\xi\eta)\xi = 0$ is immediate and hance $N^{(4)} = 0$. Finally, we have

$$N_\varphi(\varphi X, Y) - 2d\eta(\varphi X, Y)\xi = -N^{(2)}(X, Y)\xi$$

which simplifies to $N^{(2)} = 0$.

If the structure is paracontact we have already seen that $N^{(4)}(X) = (\mathcal{L}_\xi\eta)X = 2d\eta(\xi, X) = 0$. Now $N^{(2)}$ can be written

$$N^{(2)}(X, Y) = 2d\eta(\varphi X, Y) + 2d\eta(X, \varphi Y) = 2g(\varphi X, \varphi Y) - 2g(\varphi Y, \varphi X) = 0.$$

Turning to $N^{(3)}$, since $d\eta$ is invariant under the action of ξ , we have

$$\begin{aligned}
0 &= (\mathcal{L}_\xi d\eta)(X, Y) = \xi g(X, \varphi Y) - g([\xi, X], \varphi Y) - g(X, \varphi[\xi, Y]) = \\
&= (\mathcal{L}_\xi g)(X, Y) + g(X, N^{(3)}(Y)).
\end{aligned}$$

□

A paracontact structure for which ξ is Killing vector field is called a *K-paracontact structure*.

Proposition 2.4. *For an almost paracontact metric structure (φ, ξ, η, g) , the covariant derivative $\nabla\varphi$ of φ with respect to the Levi-Civita connection ∇ is given by*

$$(2.5) \quad 2g((\nabla_X\varphi)Y, Z) = -dF(X, Y, Z) - dF(X, \varphi Y, \varphi Z) - N^{(1)}(Y, Z, \varphi X) \\ + N^{(2)}(Y, Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z).$$

For a paracontact metric structure (φ, ξ, η, g) , the formula (2.5) simplifies to

$$(2.6) \quad 2g((\nabla_X\varphi)Y, Z) = -N^{(1)}(Y, Z, \varphi X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z)$$

Proof. The Levi-Civita connection ∇ with respect to g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) + \\ + g([Z, X], Y) - g([Y, Z], X).$$

On the other hand, dF can be expressed in the following way

$$dF(X, Y, Z) = XF(Y, Z) + YF(Z, X) + ZF(X, Y) - F([X, Y], Z) - \\ - F([Z, X], Y) - F([Y, Z], X).$$

The last two equations imply (2.5). The equation (2.6) follows from equation (2.5) and equalities $N^{(2)} = 0$ and $F = d\eta$. \square

We have seen that on a contact manifold, $N^{(3)}$ vanishes if and only if ξ is Killing (*Proposition 2.3*) For a general paracontact structure the tensor field $N^{(3)}$ encodes many important properties and for simplicity we define a tensor field h on a paracontact manifold by

$$h = \frac{1}{2}\mathcal{L}_\xi\varphi = \frac{1}{2}N^{(3)}.$$

Lemma 2.5. *On a paracontact metric manifold, h is a symmetric operator,*

$$(2.7) \quad \nabla_X\xi = -\varphi X + \varphi hX,$$

h anti-commutes with φ and $\text{tr}h = h\xi = 0$.

Proof. We have already seen that on a paracontact metric manifold, $\nabla_\xi\varphi = 0$, $\nabla_\xi\xi = 0$ and $N^{(2)} = 0$. Thus

$$-g((\mathcal{L}_\xi\varphi)X, Y) + \eta(\nabla_X\varphi Y) + \eta(\nabla_{\varphi X}Y) - \eta([\varphi X, Y]) = \\ = g(\nabla_{\varphi X}\xi, Y) + \eta(\nabla_{\varphi X}Y) - \eta([\varphi X, Y]) = (\mathcal{L}_{\varphi X}\eta)Y = (\mathcal{L}_{\varphi Y}\eta)X = \\ -g((\mathcal{L}_\xi\varphi)Y, X) + \eta(\nabla_Y\varphi X) + \eta(\nabla_{\varphi Y}X) - \eta([\varphi Y, X]).$$

Hence $g((\mathcal{L}_\xi\varphi)X, Y) = g((\mathcal{L}_\xi\varphi)Y, X)$.

For the second statement, using *Proposition 2.4*, we have

$$2g((\nabla_X\varphi)\xi, Z) = -g(N^{(1)}(\xi, Z), \varphi X) - 2d\eta(\varphi Z, X) = -g((\mathcal{L}_\xi\varphi)Z, X) + \\ + 2g(Z, X) - 2\eta(X)\eta(Z) = -g((\mathcal{L}_\xi\varphi)X, Z) + 2g(Z, X) - 2\eta(X)\eta(Z)$$

and hence $\varphi\nabla_X\xi = hX - X + \eta(X)\xi$. Applying φ we obtain

$$\nabla_X\xi = -\varphi X + \varphi hX.$$

To see the anti-commutativity, note that

$$2g(X, \varphi Y) = 2d\eta(X, Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = -g(\varphi X, Y) + g(\varphi hX, Y) + g(\varphi Y, X) - g(\varphi hY, X).$$

Therefore $0 = g(\varphi hX, Y) + g(Y, h\varphi X)$ giving $h\varphi + \varphi h = 0$. From the equality $\varphi \nabla_X \xi = hX - X + \eta(X)\xi$ we get $h\xi = 0$. \square

Corollary 2.6. *On a paracontact manifold, $\delta\eta = 0$, where δ is the co-differential.*

Lemma 2.7. *On a paracontact metric manifold we have the formula*

$$(2.8) \quad (\nabla_{\varphi X} \varphi)Y - (\nabla_X \varphi)Y = 2g(X, Y)\xi - (X - hX + \eta(X)\xi)\eta(Y)$$

Proof. Either using (2.6) or by direct differentiation of $\nabla_Y \xi = -\varphi Y + \varphi hY$, we obtain

$$(2.9) \quad (\nabla_X F)(\varphi Y, Z) - (\nabla_X F)(Y, \varphi Z) = \eta(Y)g(X - hX, \varphi Z) + \eta(Z)g(X - hX, \varphi Y).$$

Replacing Z by φZ and using (2.6), we get

$$(2.10) \quad (\nabla_X F)(\varphi Y, \varphi Z) - (\nabla_X F)(Y, Z) = \eta(Y)g(X - hX, Z) - \eta(Z)g(X - hX, Y)$$

Now, since $dF = 0$ we have

$$(2.11) \quad \begin{aligned} & -(\nabla_X F)(Y, Z) - (\nabla_Y F)(Z, X) - (\nabla_Z F)(X, Y) - \\ & -(\nabla_X F)(\varphi Y, \varphi Z) - (\nabla_{\varphi Y} F)(\varphi Z, X) - (\nabla_{\varphi Z} F)(X, \varphi Y) + \\ & + (\nabla_{\varphi X} F)(\varphi Y, Z) + (\nabla_{\varphi Y} F)(Z, \varphi X) + (\nabla_Z F)(\varphi X, \varphi Y) + \\ & + (\nabla_{\varphi X} F)(Y, \varphi Z) + (\nabla_Y F)(\varphi Z, \varphi X) + (\nabla_{\varphi Z} F)(\varphi X, Y) = 0. \end{aligned}$$

Now (2.9), (2.10) and (2.11) give

$$(\nabla_{\varphi X} F)(\varphi Y, Z) - (\nabla_X F)(Y, Z) = -2g(X, Y)\eta(Z) + g(X - hX + \eta(X)\xi, Z)\eta(Y)$$

from which the result follows. \square

We recall that a *paraSasakian manifold* is a normal paracontact metric manifold.

Theorem 2.8. *An almost paracontact metric structure (φ, ξ, η, g) is paraSasakian if and only if*

$$(2.12) \quad (\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$$

In particular, a paraSasakian manifold is K-paracontact.

Proof. Suppose (φ, ξ, η, g) is paraSasakian. Then *Proposition 2.4* yields

$$2g((\nabla_X \varphi)Y, Z) = -2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z) = 2g(-g(X, Y)\xi + \eta(Y)X, Z).$$

Conversely, assume $(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$, set $Y = \xi$ to get $-\varphi \nabla_X \xi = -\eta(X)\xi + X$. Hence $\nabla_X \xi = -\varphi X$ and therefore

$$2d\eta(X, Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 2g(X, \varphi Y)$$

showing that (φ, ξ, η, g) is a paracontact metric structure.

Now, we calculate

$$\begin{aligned} N_\varphi(X, Y) - 2d\eta(X, Y)\xi &= [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] - 2d\eta(X, Y)\xi = \\ &= (\nabla_{\varphi X} \varphi)Y - (\nabla_{\varphi Y} \varphi)X - \varphi(\nabla_X \varphi)Y + \varphi(\nabla_Y \varphi)X - 2d\eta(X, Y)\xi = -g(\varphi X, Y)\xi + \end{aligned}$$

$$\begin{aligned}
& +\eta(Y)\varphi X + g(\varphi Y, X)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X + \eta(X)\varphi Y - 2d\eta(X, Y)\xi = \\
& = 2g(X, \varphi Y) - 2d\eta(X, Y)\xi = 0.
\end{aligned}$$

Therefore, (φ, ξ, η, g) is paraSasakian. \square

3. CURVATURE OF PARACONTACT MANIFOLDS

In this chapter we discuss some aspects of the curvature of paracontact manifolds. We begin with some preliminaries concerning the tensor field h .

Proposition 3.1. *On a paracontact manifold M^{2n+1} we have the formulas*

$$(3.13) \quad (\nabla_\xi h)X = -\varphi X + h^2\varphi X + \varphi R(\xi, X)\xi,$$

$$(3.14) \quad (R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = 2\varphi^2 X - 2h^2 X$$

Proof. Using Lemma 2.5, we calculate

$$R(\xi, X)\xi = \nabla_\xi(-\varphi X + \varphi hX) + \varphi[\xi, X] - \varphi h[\xi, X].$$

Applying φ , taking into account that $\nabla_\xi \varphi = 0$, we obtain

$$\varphi R(\xi, X)\xi = -\nabla_X \xi + (\nabla_\xi h)X + h\nabla_X \xi.$$

Apply Lemma 2.5 to get (3.13). Multiply (3.13) with φ to derive

$$R(\xi, X)\xi = \varphi^2 X + \varphi(\nabla_\xi h)X - h^2 X.$$

Taking into account that $\varphi R(\xi, \varphi X)\xi = \varphi^2 X - \varphi(\nabla_\xi h)X - h^2 X$, we get (3.14). \square

Corollary 3.2. *On a paracontact metric manifold M^{2n+1} the Ricci curvature in the direction of ξ is given by*

$$(3.15) \quad Ric(\xi, \xi) = -2n + |h|^2$$

On a K -paracontact metric manifold M^{2n+1} we have $Ric(\xi, \xi) = -2n$.

Proposition 3.3. *On a paraSasakian manifold*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X.$$

Proof. We calculate

$$\begin{aligned}
R(X, Y)\xi &= -\nabla_X \varphi Y + \nabla_Y \varphi X + \varphi[X, Y] = -(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = \\
&= \eta(X)Y - \eta(Y)X.
\end{aligned}$$

\square

Lemma 3.4. *The curvature tensor of a paracontact metric manifold satisfies*

$$(3.16) \quad R(\xi, X, Y, Z) = -(\nabla_X F)(Y, Z) + g(X, (\nabla_Y \varphi h)Z) - g(X, (\nabla_Z \varphi h)Y),$$

$$\begin{aligned}
(3.17) \quad & R(\xi, X, Y, Z) + R(\xi, X, \varphi Y, \varphi Z) - R(\xi, \varphi X, \varphi Y, Z) - R(\xi, \varphi X, Y, \varphi Z) \\
&= -2(\nabla_{hX} F)(Y, Z) + 2g(X - hX, Z)\eta(Y) - 2g(X - hX, Y)\eta(Z).
\end{aligned}$$

Proof. Differentiating $\nabla_Z \xi = -\varphi Z + \varphi h Z$, we obtain

$$R(Y, Z)\xi = -(\nabla_Y \varphi)Z + (\nabla_Z \varphi)Y + (\nabla_Y \varphi h)Z - (\nabla_Z \varphi h)Y$$

which, since $dF = 0$, yields the first formula (3.16). Set

$$\begin{aligned} A(X, Y, Z) &= -(\nabla_X F)(Y, Z) - (\nabla_X F)(\varphi Y, \varphi Z) + (\nabla_{\varphi X} F)(Y, \varphi Z) \\ &\quad + (\nabla_{\varphi X} F)(\varphi Y, Z) \\ B(X, Y, Z) &= g(X, (\nabla_Y \varphi h)Z) + g(X, (\nabla_{\varphi Y} \varphi h)\varphi Z) - g(\varphi X, (\nabla_Y \varphi h)\varphi Z) \\ &\quad - g(\varphi X, (\nabla_{\varphi Y} \varphi h)Z). \end{aligned}$$

Use (3.16) to see that the left hand side of (3.17) is equal to $A(X, Y, Z) + B(X, Y, Z) - B(X, Z, Y)$. The proof of *Lemma 3.4* yields

$$A(X, Y, Z) = -2g(X, Y)\eta(Z) + 2g(X, Z)\eta(Y).$$

It is straightforward to show that $\eta((\nabla_{\varphi Y} h)Z) = g(Y + hY, hZ)$. Rewrite B in the form

$$\begin{aligned} B(X, Y, Z) &= g(X, (\nabla_Y \varphi)hZ) - g(\varphi hX, (\nabla_{\varphi Y} \varphi)Z) - g(\varphi X, (\nabla_{\varphi Y} \varphi)hZ) - \\ &\quad - g(\varphi X, h(\nabla_Y \varphi)Z) + \eta(X)(\nabla_{\varphi Y} \eta)hZ. \end{aligned}$$

Use *Lemma 3.4* again to obtain

$$B(X, Y, Z) = -2g(hX, (\nabla_Y \varphi)Z) + 2g(hX, Y)\eta(Z) + 2g(hY, hZ)\eta(X).$$

Finally, compute $A(X, Y, Z) + B(X, Y, Z) - B(X, Z, Y)$, use $dF = 0$ to get the result. \square

Let us fix a local coordinates (x^1, \dots, x^{2n+1}) . We shall use the Einstein summation convention. The equations (2.1), (2.2) and (2.7), in local coordinates, have the expression

$$\begin{aligned} \eta_r \xi^r &= 1, & \varphi_r^i \xi^r &= 0, & \eta_r \varphi_j^r &= 0, & \varphi_r^i \varphi_j^r &= \delta_j^i - \xi^i \eta_j, \\ g_{rs} \varphi_j^r \varphi_k^s &= -g_{jk} + \eta_j \eta_k, & g_{jr} \xi^r &= \eta_j. \end{aligned}$$

We get using (2.6) that

$$\begin{aligned} \nabla_i \eta_j - \nabla_j \eta_i &= 2\varphi_{ij} = 2g_{ir} \varphi_j^r, \\ (3.18) \quad \nabla_r \varphi_j^r &= 2n\eta_j, \quad \xi^r \nabla_r \varphi_j^i = 0, \quad \nabla_r \eta_s \varphi_i^r \varphi_j^s = \nabla_j \eta_i, \end{aligned}$$

$$(3.19) \quad \nabla_r \eta_i \varphi_j^r \text{ and } \nabla_i \eta_r \varphi_j^r \text{ are symmetric in } i, j.$$

Moreover, *Lemma 2.5* implies

$$(3.20) \quad \nabla_i \eta_j = \varphi_{ij} + \varphi_{ir} h_j^r, \quad h_{ij} = h_{ji} = g_{jr} h_i^r, \quad \varphi_r^i h_j^r = -h_r^i \varphi_j^r, \quad h_{ij} \xi^j = 0.$$

Consequently, (3.20) yields

$$(3.21) \quad \nabla_r \eta_i \nabla^r \eta_j = -g_{ij} + \eta_i \eta_j - 2h_{ij} - h_{ir} h_j^r.$$

From the equations (3.14) and (3.15) we also have

$$\begin{aligned} (3.22) \quad R_{irsj} \xi^r \xi^s - R_{arsb} \xi^r \xi^s \varphi_i^a \varphi_j^b &= -2g_{ij} + 2\eta_i \eta_j + 2h_{ir} h_j^r, \\ Ric(\xi, \xi) &= -2n + |h|^2, \end{aligned}$$

where $|h|^2 = g^{ir}g^{js}h_{ij}h_{rs}$, for $h = (h_{ij})$.

Lemma 3.5. *Let $(M, g, \varphi, \eta, \xi)$ be a paracontact pseudo-Riemannian manifold. Then the Ricci tensor Ric of the Levi-Chevita connection satisfies the following relations:*

$$(3.23) \quad Ric_{jr}\xi^r = \nabla_r \nabla_j \xi^r = \nabla_r \nabla^r \eta_j - 4n\eta_j,$$

$$(3.24) \quad \varphi_j^s \nabla^r \nabla_r \varphi_{ks} + \varphi_k^s \nabla^r \nabla_r \varphi_{js} = 2\nabla_r \varphi_{sj} \nabla^r \varphi_k^s - Ric_{jr}\xi^r \eta_k - Ric_{kr}\xi^r \eta_j \\ + 2h_{jr}h_k^r + 4h_{jk} + 2g_{jk} - 2(4n+1)\eta_j\eta_k.$$

Proof. Contracting $R_{ijr}^k \xi^r = \nabla_i \nabla_j \xi^k - \nabla_j \nabla_i \xi^k$ with respect to i and k , we obtain the first equality in (3.23). To verify the second equality, we observe that $\nabla^r \nabla_r \eta_j = \nabla^r (2\varphi_{rj}) + \nabla^r \nabla_r \eta_r$. Then use (3.18) to get (3.23). Next, applying the hyperbolic Laplacian $\nabla^r \nabla_r$ to $\varphi_j^s \varphi_{ks} = g_{jk} - \eta_j \eta_k$, we obtain

$$\varphi_j^s \nabla^r \nabla_r \varphi_{ks} + \varphi_k^s \nabla^r \nabla_r \varphi_{js} - 2\nabla_r \varphi_{sj} \nabla^r \varphi_k^s = -\eta_k \nabla^r \nabla_r \eta_j - \eta_j \nabla^r \nabla_r \eta_k - 2\nabla_r \eta_j \nabla^r \eta_k.$$

The latter together with (3.21) and (3.23) yields (3.24). \square

The obstruction an almost paracontact pseudo-Riemannian manifold to be a paraSasakian, described in Theorem 2.8, is the tensor $P = (P_{rsi})$ defined by

$$(3.25) \quad P_{rsi} = \nabla_r \varphi_{si} - \eta_i g_{rs} + \eta_s g_{ri}.$$

Lemma 3.6. *On a paracontact metric manifold $P_{rsi}P_j^{rs}$ is given by*

$$(3.26) \quad P_{rsi}P_j^{rs} = \nabla_r \varphi_{si} \nabla^r \varphi_j^s + 2h_{ij} - g_{ij} - (2n-1)\eta_i \eta_j.$$

Proof. First we get

$$P_{rsi}P_j^{rs} = \nabla_r \varphi_{si} \nabla^r \varphi_j^s + \eta_s \nabla_i \varphi_j^s + \eta_s \nabla_j \varphi_i^s + g_{ij} - (2n+1)\eta_i \eta_j.$$

Since $\eta_s \nabla_i \varphi_j^s = -\varphi_j^s \nabla_i \eta_s$, applying (3.20) to the last equation, we obtain (3.26). \square

We define the *-Ricci tensor Ric_{ij}^* and the *-scalar curvature $scal^*$ by

$$Ric_{ij}^* = g^{ps} R_{pilk} \varphi_j^l \varphi_i^k, \quad scal^* = g^{ij} Ric_{ij}^*.$$

Lemma 3.7. *The symmetric part of the *-Ricci tensor is given by*

$$(3.27) \quad Ric_{ij}^* + Ric_{ji}^* = -Ric_{ij} + Ric_{rs} \varphi_i^r \varphi_j^s - 2(2n-1)g_{ij} + \\ + 2(n-1)\eta_i \eta_j + P_{rsi}P_j^{rs} + h_{ir}h_j^r.$$

Proof. By the Ricci identity for φ , we obtain

$$(3.28) \quad \nabla_l \nabla_k \varphi_j^i - \nabla_k \nabla_l \varphi_j^i = R_{lka}^i \varphi_j^a - R_{lkj}^a \varphi_s^i.$$

Contracting the last equation with respect to i and k , we get

$$(3.29) \quad 2n \nabla_l \eta_j - \nabla_i \nabla_l \varphi_j^i = -Ric_{la} \varphi_j^a - R_{ilj}^a \varphi_s^i.$$

Transvecting (3.29) by φ_k^l , we obtain

$$(3.30) \quad 2n \nabla_l \eta_j \varphi_k^l - \varphi_k^l \nabla_i \nabla_l \varphi_j^i = -Ric_{la} \varphi_j^a \varphi_k^l + Ric_{jk}^*.$$

Transvecting (3.29) by $-\varphi_k^j$, we obtain

$$(3.31) \quad -2n \nabla_l \eta_j \varphi_k^j + \varphi_k^j \nabla_i \nabla_l \varphi_j^i = Ric_{lk} - Ric_{la} \xi^a \eta_k + Ric_{lk}^*.$$

Change l to j in (3.31). Then the obtained result and (3.30) imply

$$4n\varphi_{rj}\varphi_k^r - \varphi_k^r\nabla^i(\nabla_r\varphi_{ij} - \nabla_j\varphi_{ir}) = Ric_{jk} - Ric_{rs}\varphi_j^r\varphi_k^s - Ric_{js}\xi^s\eta_k + 2Ric_{jk}^*.$$

Since $\nabla_r\varphi_{ij} + \nabla_i\varphi_{jr} + \nabla_j\varphi_{ri} = 0$, the above is written as

$$-4n(g_{kj} - \eta_k\eta_j) + \varphi_k^r\nabla^i\nabla_i\varphi_{jr} = Ric_{jk} - Ric_{rs}\varphi_j^r\varphi_k^s - Ric_{js}\xi^s\eta_k + 2Ric_{jk}^*.$$

Take the symmetric part of the latter equation, use (3.24) and (3.26) to derive (3.27). \square

We define $P(X) = (P_{rsi}X^i)$. Then we get $|P(X)|^2 = (P_{rsi}P_j^rX^iX^j)$. By (3.26) it easy to verify

$$(3.32) \quad |P(\xi)|^2 = |h|^2.$$

Therefore, if (M, φ, η, g) is a K-paracontact manifold, then $|P(\xi)|^2 = 0$.

By Lemma 3.7 we obtain the following

Corollary 3.8. *If a paracontact manifold (M, φ, η, g) is a paraSasakian, then*

$$(3.33) \quad Ric_{ij}^* + Ric_{ji}^* = -Ric_{ij} + Ric_{rs}\varphi_i^r\varphi_j^s - 2(2n-1)g_{ij} + 2(n-1)\eta_i\eta_j$$

The equalities (3.27) and (3.15) give

Corollary 3.9. *Let (M, φ, η, g) be a paracontact manifold. Then*

$$(3.34) \quad scal + scal^* + 4n^2 = |h|^2 + \frac{1}{2}|\nabla\varphi|^2 - 2n,$$

where $|P|^2 = |\nabla\varphi|^2 - 4n$. If (M, φ, η, g) is paraSasakian manifold, then

$$scal + scal^* + 4n^2 = 0.$$

In the contact case the identity (3.34) has been proven by Olszak ([7], see also [10]).

Theorem 3.10. *Let (M, φ, η, g) be a locally conformally equivalent to a flat paracontact manifold of dimension $2n+1 \geq 5$. For any unit X orthogonal to ξ*

$$(3.35) \quad Ric(X, X) - Ric(\varphi X, \varphi X) = -4n - \frac{1}{n(2n-3)}(2n(2n+1) + scal) + \frac{2n-1}{2n-3}(|P(X)|^2 + |h(X)|^2).$$

If (M, φ, η, g) is a conformally flat paraSasakian manifold and $2n+1 \geq 5$, then

$$Ric(X, X) - Ric(\varphi X, \varphi X) = -4n - \frac{1}{n(2n-3)}(2n(2n+1) + scal).$$

Proof. Recall that a Riemannian manifold is locally conformally flat exactly when the Weyl curvature vanishes due to the Weyl's theorem. Let (M, φ, η, g) be a conformally flat paracontact manifold. Then the Riemannian curvature tensor R is expressed as

$$R_{ijkl} = \frac{1}{2n-1}(Ric_{jk}g_{il} - Ric_{ik}g_{jl} - Ric_{jl}g_{ik} + Ric_{il}g_{jk}) - \frac{scal}{2n(2n-1)}(g_{jk}g_{il} - g_{ik}g_{jl})$$

Hence, $Ric^*(X, X)$ for any unit $X \perp \xi$ is given by

$$Ric^*(X, X) = -\frac{1}{2n-1}(Ric(X, X) - Ric(\varphi X, \varphi X)) + \frac{scal}{2n(2n-1)}$$

On the other hand, (3.27) gives

$$2Ric^*(X, X) = -Ric(X, X) + Ric(\varphi X, \varphi X) - 2(2n-1) + |P(X)|^2 + |h(X)|^2.$$

Combining the last two equations we obtain (3.35). \square

Remark 3.11. Let $(e_i, \varphi e_i, \xi)$ be an adapted basis of a conformally flat paracontact manifold. Then, using (3.15) and (3.35), we can show that the scalar curvature $scal$ is given by

$$scal = -2n(2n+1) + \frac{2n-1}{4(n-1)(2n-3)}|P|^2 + \frac{2n-3}{2(n-1)}|h|^2.$$

Theorem 3.12. *If a paracontact manifold M^{2n+1} is of constant sectional curvature c and dimension $2n+1 \geq 5$, then $c=-1$ and $|h|^2 = 0$.*

Proof. Recall from Proposition 3.1 that, $\frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = \varphi^2 X - h^2 X$; thus if $R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$, then $c(\eta(X)\xi - X - \varphi^2 X) = 2\varphi^2 X - 2h^2 X$. Therefore $h^2 X = (c+1)\varphi^2 X$ and hence $|h|^2 = 2n(c+1)$. Now from Lemma 3.4

$$\begin{aligned} (\nabla_{hX} F)(Y, Z) &= -(c+1)g(X, Y)\eta(Z) + (c+1)g(X, Z)\eta(Y) + g(hX, Y)\eta(Z) \\ &\quad - g(hX, Z)\eta(Y). \end{aligned}$$

Replacing X by hX , we have

$$\begin{aligned} (\nabla_{h^2 X} F)(Y, Z) &= -(c+1)g(hX, Y)\eta(Z) + (c+1)g(hX, Z)\eta(Y) + g(h^2 X, Y)\eta(Z) \\ &\quad - g(h^2 X, Z)\eta(Y). \end{aligned}$$

Hence, $(c+1)((\nabla_X \varphi)Y + g(X - hX, Y)\xi - (X - hX)\eta(Y)) = 0$. We have two cases

CASE 1. If $c = -1$ then we have $|h|^2 = 0$.

CASE 2. If $c \neq -1$ then $(\nabla_X \varphi)Y = -g(X - hX, Y)\xi + (X - hX)\eta(Y)$. Using the latter, we compute $|\nabla \varphi|^2$ and applying $|h|^2 = 2n(c+1)$, we obtain $|\nabla \varphi|^2 = 4n(c+2)$. On the other hand $scal = 2n(2n+1)c$ and $scal^* = -2nc$ as is easily checked. Now from the formula in Corollary 3.9, we obtain $4n^2(c+1) = 4n(c+1)$. This is a contradiction, because $n > 1$ and $c \neq -1$. \square

We restrict our attention to paraSasakian manifolds. We begin with

Lemma 3.13. *On a paraSasakian manifold we have*

$$(3.36) \quad \begin{aligned} R(X, Y, \varphi Z, W) + R(X, Y, Z, \varphi W) &= -d\eta(X, W)g(Y, Z) + d\eta(X, Z)g(Y, W) \\ &\quad -d\eta(Y, Z)g(X, W) + d\eta(Y, W)g(X, Z), \end{aligned}$$

$$(3.37) \quad \begin{aligned} R(\varphi X, \varphi Y, \varphi Z, \varphi W) - R(X, Y, Z, W) &= \eta(X)\eta(W)g(Y, Z) \\ &\quad + \eta(Y)\eta(Z)g(X, W) - \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(Z)g(Y, W), \end{aligned}$$

$$(3.38) \quad \begin{aligned} R(X, \varphi X, Y, \varphi Y) &= -R(X, Y, X, Y) + R(X, \varphi Y, X, \varphi Y) \\ &\quad + \eta(X)\eta(Y)g(X, Y) + 2(d\eta(X, Y)d\eta(X, Y) - g(X, Y)g(X, Y) + |X|^2|Y|^2). \end{aligned}$$

$$(3.39) \quad Ric(X, \varphi Y) + Ric(\varphi X, Y) = -d\eta(X, Y)$$

Proof. The first equality follows by definition and $(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$. Using the first equality we obtain the second. The third equality follows by the first Bianchi identity to $R(X, \varphi X, Y, \varphi Y)$ and using the first equality. Finally choosing a φ -basis and second equality we obtain the fourth equality. \square

Corollary 3.14. *On a paraSasakian manifold for X, Y, Z, W orthogonal to ξ we have*

$$\begin{aligned} R(\varphi X, \varphi Y, \varphi Z, \varphi W) &= R(X, Y, Z, W); \\ R(X, \varphi X, Y, \varphi Y) &= -R(X, Y, X, Y) + R(X, \varphi Y, X, \varphi Y) + \\ &\quad + 2(d\eta(X, Y)d\eta(X, Y) - g(X, Y)g(X, Y) + |X|^2|Y|^2); \\ Ric(X, \varphi Y) + Ric(\varphi X, Y) &= 0. \end{aligned}$$

The Bianchi identities and equation (3.37) yield

Lemma 3.15. *The Ricci tensor Ric of a $(2n+1)$ -dimensional paraSasakian manifold M satisfies the relations*

$$\begin{aligned} Ric(X, Y) &= \frac{1}{2} \sum_{i=1}^{2n+1} R(X, \varphi Y, e_i, \varphi e_i) - (2n-1)g(X, Y) - \eta(X)\eta(Y), \\ Ric(\varphi X, \varphi Y) &= -Ric(X, Y) - 2n\eta(X)\eta(Y), \\ (\nabla_Z Ric)(X, Y) &= (\nabla_X Ric)(Y, Z) - (\nabla_{\varphi Y} Ric)(\varphi X, Z) - \eta(X)Ric(\varphi Y, Z) \\ &\quad - 2\eta(Y)Ric(\varphi X, Z) - 2n\eta(X)g(\varphi Y, Z) - 4n\eta(Y)g(\varphi X, Z). \end{aligned}$$

4. CANONICAL PARACONTACT CONNECTION AND CONFORMAL (GAUGE) TRANSFORMATION

Let $(M, \varphi, \xi, \eta, g)$ be a paracontact manifold. All paracontact forms $\tilde{\eta}$ generating the same horizontal distribution $\mathbb{D} = Ker \eta$ are connected by $\tilde{\eta} = \sigma\eta$ for a positive smooth function σ on M . We consider another paracontact form $\tilde{\eta} = \sigma\eta$ and define structure tensors $(\tilde{\varphi}, \tilde{\xi}, \tilde{g})$ corresponding to $\tilde{\eta}$ using the condition:

(\star) For each point x of M , the actions of φ and $\tilde{\varphi}$ are identical on \mathbb{D}_x

By calculating $d\tilde{\eta} = d(\sigma\eta)$, we obtain

$$(4.40) \quad 2\tilde{\varphi}_{ij} = \sigma_i\eta_j - \sigma_j\eta_i + 2\sigma\varphi_{ij},$$

where $\sigma_i = \nabla_i \sigma$. By $\tilde{\xi}^i \tilde{\varphi}_{ij} = 0$, $\tilde{\eta}_i \tilde{\xi}^i = 1$ and (4.40), we obtain $\tilde{\xi} \sigma = \frac{1}{\sigma} \xi \sigma$, and

$$(4.41) \quad \tilde{\xi}^k = \frac{1}{\sigma} \xi^k - \frac{1}{2\sigma^2} \varphi_j^k \sigma^j.$$

So we define ζ by $\zeta^k = -\frac{1}{2\sigma} \varphi_j^k \sigma^j$ and get

$$\tilde{\xi}^k = \frac{1}{\sigma} (\xi^k + \zeta^k).$$

By $\tilde{\varphi}_{ij} \tilde{\varphi}^{jk} = \delta_i^k - \tilde{\xi}^k \tilde{\eta}_i$ and $\tilde{\eta}_j \tilde{\varphi}^{jk} = 0$, $\tilde{\varphi}^{jk}$ is determined:

$$(4.42) \quad \tilde{\varphi}^{jk} = \frac{1}{\sigma} \varphi^{jk}.$$

Now, by the condition (\star) we can put $\tilde{\varphi}_j^i = \varphi_j^i + v^i \eta_j$ for some vector field v on M .

By $\tilde{\eta}_i \tilde{\varphi}_j^i = 0$ and $\tilde{\varphi}_j^i \tilde{\varphi}_k^j = \delta_k^i - \tilde{\xi}^i \tilde{\eta}_k$, v is determined:

$$v^i = \frac{1}{2\sigma} (\sigma^i - \xi \sigma \cdot \xi^i).$$

By the expressions of $\tilde{\varphi}_{ij}$ and $\tilde{\varphi}_j^i$, we obtain

$$\tilde{g}_{ij} = \sigma(g_{ij} - \eta_i \zeta_j - \eta_j \zeta_i) + \sigma(\sigma - 1 + |\zeta|^2) \eta_i \eta_j.$$

The inverse matrix (\tilde{g}^{jk}) of (\tilde{g}_{ij}) is given by

$$\tilde{g}^{jk} = \frac{1}{\sigma} (g^{jk} - \xi^j \xi^k) + \frac{1}{\sigma^2} (\xi^j + \zeta^j) (\xi^k + \zeta^k).$$

The last relation can be rewritten as

$$(4.43) \quad \sigma(\tilde{g}^{jk} - \tilde{\xi}^j \tilde{\xi}^k) = g^{jk} - \xi^j \xi^k.$$

Summarizing the above discussions, we obtain

Lemma 4.1. *Under condition (\star) , a gauge transformation $\eta \rightarrow \tilde{\eta} = \sigma \eta$ of a paracontact form η induces the transformation of the structure tensors of the form:*

$$\tilde{\xi}^k = \frac{1}{\sigma} (\xi^k + \zeta^k), \quad \zeta^k = -\frac{1}{2\sigma} \varphi_j^k \sigma^j,$$

$$\tilde{\varphi}_j^i = \varphi_j^i + \frac{1}{2\sigma} (\sigma^i - \xi \sigma \cdot \xi^i) \eta_j,$$

$$\tilde{g}_{ij} = \sigma(g_{ij} - \eta_i \zeta_j - \eta_j \zeta_i) + \sigma(\sigma - 1 + |\zeta|^2) \eta_i \eta_j.$$

We call the transformation of the structure tensors given by Lemma 4.1 a *gauge (conformal) transformation of paracontact pseudo-Riemannian structure*. When σ is constant this is a \mathbb{D} -homothetic transformation studied in the Subsection 4.1

On a strongly pseudo-convex CR-manifold Tanaka [9] and Webster [12] introduced a canonical connection preserving the structure called *Tanaka-Webster connection*. Tanno generalized this connection extending its definition to the general contact metric manifold.

Following [11], we consider the connection $\tilde{\nabla}$ defined by

$$(4.44) \quad \begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \eta(X) \varphi Y - \eta(Y) \nabla_X \xi + (\nabla_X \eta) Y \cdot \xi \\ &= \nabla_X Y + \eta(X) \varphi Y + \eta(Y) (\varphi X - \varphi h X) + g(X, \varphi Y) \cdot \xi - g(h X, \varphi Y) \cdot \xi. \end{aligned}$$

The torsion of this connection is then

$$(4.45) \quad \begin{aligned} T(X, Y) &= \eta(X)\varphi Y - \eta(Y)\varphi X - \eta(Y)\nabla_X \xi + \eta(X)\nabla_Y \xi + 2d\eta(X, Y)\xi \\ &= \eta(X)\varphi hY - \eta(Y)\varphi hX + 2g(X, \varphi Y)\xi. \end{aligned}$$

Proposition 4.2. *On a paracontact manifold the connection $\tilde{\nabla}$ has the properties*

$$(4.46) \quad \begin{aligned} \tilde{\nabla}\eta &= 0, \quad \tilde{\nabla}\xi = 0, \quad \tilde{\nabla}g = 0, \\ (\tilde{\nabla}_X \varphi)Y &= (\nabla_X \varphi)Y + g(X - hX, Y)\xi - \eta(Y)(X - hX), \\ T(\xi, \varphi Y) &= -\varphi T(\xi, Y), \quad Y \in \Gamma(\mathbb{D}) \quad \text{or} \quad Y \in \Gamma(TM) \\ T(X, Y) &= 2d\eta(X, Y)\xi, \quad X, Y \in \Gamma(\mathbb{D}). \end{aligned}$$

Proof. Calculation is straightforward by using (4.44) and (4.45). \square

Definition 4.3. We call the connection $\tilde{\nabla}$ defined above on a paracontact manifold the *canonical paracontact connection*.

We calculate the curvature of $\tilde{\nabla}$. Let W be the $(1, 2)$ -tensor field expressing the difference between $\tilde{\nabla}$ and ∇ , $W_{ij}^k = \tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$. We obtain using (4.44) that

$$(4.47) \quad \begin{aligned} \tilde{R}_{ijk}^l &= R_{ijk}^l + \nabla_i \varphi_k^l \eta_j - \nabla_j \varphi_k^l \eta_i + 2\varphi_{ij} \varphi_k^l - \varphi_s^l \nabla_j \xi^s \eta_i \eta_k + \varphi_s^l \nabla_i \xi^s \eta_j \eta_k + \\ &\quad + \xi^l \nabla_i \eta_s \varphi_k^s \eta_j - \xi^l \nabla_j \eta_s \varphi_k^s \eta_i - \xi^l R_{ijs}^s \eta_s - \eta_k R_{ijs}^s \xi^s + \nabla_j \eta_k \nabla_i \xi^l - \nabla_i \eta_k \nabla_j \xi^l. \end{aligned}$$

Contracting (4.47) with respect to i and l , we obtain

$$\widetilde{Ric}_{jk} = Ric_{jk} - 2g_{jk} + 2\eta_j \eta_k - \eta_k Ric_{js} \xi^s - R_{jsrk} \xi^s \xi^r - \nabla_r \eta_k \nabla_j \xi^r.$$

Since $\widetilde{Ric}_{jk} \xi^j \xi^k = 0$, we define the scalar curvature of the canonical paracontact connection of a paracontact pseudo-Riemannian manifold (M, ξ, η, g) by $W_1 = g^{jk} \widetilde{Ric}_{jk}$. Using (3.23), we obtain $\nabla_r \eta_s \nabla^s \xi^r = -Ric_{rs} \xi^r \xi^s$. Hence,

$$(4.48) \quad W_1 = scal - Ric(\xi, \xi) - 4n.$$

Let f and f' be two functions on a paracontact pseudo-Riemannian manifold (M, ξ, η, g) . We define operator $\Delta_{\mathbb{D}}$ acting on the space of functions by using the hyperbolic Laplacian Δ and ξ :

$$\Delta_{\mathbb{D}} f = \Delta f - \xi \xi f = (g^{ij} - \xi^i \xi^j) \nabla_i \nabla_j f,$$

and $(df; df')_{\mathbb{D}}$ by

$$(df; df')_{\mathbb{D}} = (g^{ij} - \xi^i \xi^j) \nabla_i f \nabla_j f'.$$

Furthermore, $|df|_{\mathbb{D}}^2$ means $(df; df)_{\mathbb{D}}$, which is equal to $|df|^2 - (\xi f)^2$.

Theorem 4.4. *Let $(\eta, g) \rightarrow (\tilde{\eta} = \sigma\eta, \tilde{g})$ be a conformal (gauge) transformation of paracontact pseudo-Riemannian structure. Then the transformation of the scalar curvature W_1 of the canonical paracontact connection is given by*

$$(4.49) \quad \sigma \widetilde{W}_1 = W_1 - \frac{2(n+1)}{\sigma} \Delta_{\mathbb{D}} \sigma - \frac{(n+1)(n-2)}{\sigma^2} |d\sigma|_{\mathbb{D}}^2.$$

Proof. We follow the scheme in [11]. Geometric quantities corresponding to \tilde{g} are denoted by \sim . We define \widetilde{W}_{jk}^i by

$$\widetilde{W}_{jk}^i = \widetilde{\Gamma}_{jk}^i - \Gamma_{jk}^i.$$

Then

$$\widetilde{W}_{jk}^i = \frac{1}{2}\tilde{g}^{ia}(\nabla_j \tilde{g}_{ak} + \nabla_k \tilde{g}_{aj} - \nabla_a \tilde{g}_{jk}).$$

The Ricci tensor \widetilde{Ric} is given by

$$\widetilde{Ric}_{jl} = Ric_{jl} + \nabla_r \widetilde{W}_{lj}^r - \nabla_l \widetilde{W}_{rj}^r + \widetilde{W}_{lj}^s \widetilde{W}_{rs}^r - \widetilde{W}_{rj}^s \widetilde{W}_{ls}^r.$$

Transvecting the last equality by $g^{jl} - \xi^j \xi^l$ and using (4.43) we obtain

$$(4.50) \quad \sigma(\widetilde{scal} - \widetilde{Ric}_{jl} \xi^j \xi^l) = scal - Ric_{jl} \xi^j \xi^l + (g^{jl} - \xi^j \xi^l) \nabla_r \widetilde{W}_{lj}^r - (g^{jl} - \xi^j \xi^l) \nabla_l \widetilde{W}_{rj}^r + (g^{jl} - \xi^j \xi^l) \widetilde{W}_{lj}^s \widetilde{W}_{rs}^r - (g^{jl} - \xi^j \xi^l) \widetilde{W}_{rj}^s \widetilde{W}_{ls}^r.$$

First we calculate the following:

$$\begin{aligned} \widetilde{W}_{lj}^r (g^{jl} - \xi^j \xi^l) &= \frac{1}{2} \tilde{g}^{ra} (\nabla_l \tilde{g}_{aj} + \nabla_j \tilde{g}_{al} - \nabla_a \tilde{g}_{jl}) \sigma (\tilde{g}^{jl} - \tilde{\xi}^j \tilde{\xi}^l) = \\ &= \sigma \tilde{g}^{ra} [\nabla_j (\tilde{\xi}^j \tilde{\eta}_a) - \nabla_j (\frac{1}{\sigma} (g^{jl} - \xi^j \xi^l)) \tilde{g}_{al}] - \frac{1}{2} \sigma \tilde{g}^{ra} \tilde{g}_{jl} \nabla (\frac{1}{\sigma} (g^{jl} - \xi^j \xi^l)). \end{aligned}$$

After some calculation, we obtain

$$(4.51) \quad \widetilde{W}_{lj}^r (g^{jl} - \xi^j \xi^l) = \frac{n}{\sigma} \xi \sigma \cdot \xi^r - \frac{n}{\sigma} \sigma^r,$$

$$(4.52) \quad \nabla_r (\widetilde{W}_{lj}^r (g^{jl} - \xi^j \xi^l)) = \frac{n}{\sigma^2} |d\sigma|_{\mathbb{D}}^2 - \frac{n}{\sigma} \Delta_{\mathbb{D}} \sigma.$$

Next, using $2\nabla_r (\xi^j \xi^l) \widetilde{W}_{lj}^r = \nabla_r \xi^j \xi^l \tilde{g}^{ra} (\nabla_l \tilde{g}_{aj} + \nabla_j \tilde{g}_{al} - \nabla_a \tilde{g}_{jl})$, we derive

$$(4.53) \quad \begin{aligned} \nabla_r (\xi^j \xi^l) \widetilde{W}_{lj}^r &= 4n(\sigma - 1 + |\zeta|^2) + \frac{1}{\sigma} (\sigma^r \zeta^s + \sigma^s \zeta^r) \nabla_s \eta_r + \frac{1}{\sigma^2} \xi \sigma \zeta^r \zeta^s \nabla_s \eta_r + \\ &+ \frac{1}{\sigma} \zeta^r \nabla_r \eta_j \zeta^a \nabla^j \eta_a - 2\varphi_{rj} \nabla^j \zeta^r + \frac{1}{\sigma} \zeta^r \nabla_r \eta_j (\nabla^j \zeta_a + \nabla_a \zeta^j) \zeta^a + \frac{1}{\sigma^2} (1 - |\zeta|^2) \sigma^j \zeta^r \nabla_r \eta_j. \end{aligned}$$

By a direct calculation we get

$$(4.54) \quad \tilde{g}^{ra} \nabla_j \tilde{g}_{ra} = \frac{2(n+1)}{\sigma} \sigma_j$$

Therefore

$$(4.55) \quad (g^{jl} - \xi^j \xi^l) \nabla_l \widetilde{W}_{rj}^r = \frac{(n+1)}{\sigma} \Delta_{\mathbb{D}} \sigma - \frac{(n+1)}{\sigma^2} |d\sigma|_{\mathbb{D}}^2.$$

The fifth term of the right-hand side of (4.50) is

$$(4.56) \quad (g^{jl} - \xi^j \xi^l) \widetilde{W}_{lj}^s \widetilde{W}_{rs}^r = -\frac{n(n+1)}{\sigma^2} |d\sigma|_{\mathbb{D}}^2$$

due to (4.51) and (4.54).

The sixth term of right-hand side of (4.50) is

$$(g^{jl} - \xi^j \xi^l) \widetilde{W}_{rj}^s \widetilde{W}_{ls}^r = \frac{1}{4} (g^{jl} - \xi^j \xi^l) [-\nabla_j \tilde{g}_{rs} \nabla_l \tilde{g}^{rs} - 2\nabla_a \tilde{g}_{jr} \nabla_s \tilde{g}_{lb} \tilde{g}^{sa} \tilde{g}^{rb} + 2\nabla_a \tilde{g}_{jr} \nabla_b \tilde{g}_{ls} \tilde{g}^{sa} \tilde{g}^{rb}].$$

The right-hand side of the last equality is calculated as follows:

$$\begin{aligned} \frac{1}{4}(g^{jl} - \xi^j \xi^l) \nabla_j \tilde{g}_{rs} \nabla_l \tilde{g}^{rs} &= -\frac{n+2}{2\sigma^2} |d\sigma|_{\mathbb{D}}^2 + \frac{1}{2} (2 - \sigma - \frac{1}{\sigma} - |\zeta|^2) |\nabla \xi|^2 + (1 - \frac{1}{\sigma}) (\nabla \xi; \nabla \zeta) + \\ &+ \frac{1}{\sigma} \xi^r \sigma^s \nabla_s \zeta_r - \frac{1}{2\sigma} |\nabla \zeta|^2 + \frac{1}{\sigma} \zeta^r \nabla_l \zeta_r \eta_s \nabla^l \zeta^s - \frac{1}{2\sigma} (\sigma + 1 + |\zeta|^2) \xi^r \nabla_l \zeta_r \eta_s \nabla^l \zeta^s + \frac{1}{2\sigma} |\nabla \xi \zeta|^2, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(g^{jl} - \xi^j \xi^l) \nabla_a \tilde{g}_{jr} \nabla_s \tilde{g}_{lb} \tilde{g}^{sa} \tilde{g}^{rb} &= \frac{n}{\sigma^2} [|d\sigma|_{\mathbb{D}}^2 + (\frac{1}{\sigma} - 1)(\xi\sigma)^2] + (\frac{1}{2\sigma}(\sigma - 1)^2 + |\zeta|^2) |\nabla \xi|^2 + \\ &+ \frac{1}{\sigma} (1 - \sigma) (\nabla \xi; \nabla \zeta) + (\frac{1}{2\sigma^2}(\sigma - 1)^2 + \frac{1}{\sigma} |\zeta|^2) |\nabla \zeta \xi|^2 + \frac{1}{2\sigma} |\nabla \zeta|^2 + \frac{1}{\sigma^2} (1 - \sigma) \zeta^a \nabla_a \eta_r (\xi^b + \zeta^b) \nabla_b \zeta^r - \\ &- \frac{2}{\sigma} \zeta^r \sigma^s \nabla_s \eta_r - \frac{2}{\sigma^2} \xi \sigma \cdot \zeta^r \zeta^s \nabla_s \eta_r + \frac{1}{\sigma} \zeta^s \nabla_a \eta_s \zeta^l \nabla^a \zeta_l + \frac{1}{2\sigma} (1 + |\zeta|^2) \zeta^r \nabla_a \eta_r \zeta^s \nabla^a \eta_s + \\ &+ \frac{1}{\sigma^2} \zeta^s \zeta^a \nabla_a \eta_s \zeta^l (\xi^b + \zeta^b) \nabla_b \zeta_l + \frac{1}{2\sigma^2} (1 + |\zeta|^2) (\zeta^r \zeta^s \nabla_s \eta_r)^2, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(g^{jl} - \xi^j \xi^l) \nabla_a \tilde{g}_{jr} \nabla_b \tilde{g}_{ls} \tilde{g}^{sa} \tilde{g}^{rb} &= -\frac{1}{2\sigma^2} |d\sigma|_{\mathbb{D}}^2 + \frac{1}{\sigma} \sigma^r \zeta^s \nabla_s \eta_r + \frac{1}{\sigma^2} \xi \sigma (\nabla_r \zeta^r + \zeta^r \zeta^s \nabla_s \eta_r) - \\ &- \frac{1}{2} |\zeta|^2 \nabla_b \eta_a \nabla^a \xi^b + \frac{1}{\sigma} (\nabla \zeta \xi; \nabla \xi \zeta) - \frac{1}{\sigma} (1 + |\zeta|^2) \zeta^b \nabla_b \eta_a \zeta^s \nabla^a \eta_s - \frac{1}{\sigma} \zeta^b \nabla_b \eta_a \zeta^r \nabla^a \zeta_r - \\ &- \frac{1}{2\sigma^2} |\nabla \zeta \xi + \nabla \xi \zeta + \nabla \zeta \zeta|^2 - \frac{1}{2\sigma^2} (1 + |\zeta|^2) (\zeta^r \zeta^s \nabla_s \eta_r)^2 - \frac{1}{\sigma^2} \zeta^r \zeta^a \nabla_a \eta_r \zeta^s (\xi^b + \zeta^b) \nabla_b \zeta_s. \end{aligned}$$

Since $\nabla_r \zeta^r = -\frac{n}{\sigma} \xi \sigma$, we obtain

$$\begin{aligned} (4.57) \quad (g^{jl} - \xi^j \xi^l) \tilde{W}_{rj}^s \tilde{W}_{ls}^r &= -\frac{n-1}{2\sigma^2} |d\sigma|_{\mathbb{D}}^2 - |\zeta|^2 \varphi_{rs} \nabla^s \xi^r + \frac{1}{2} \zeta^r \nabla_l \eta_r \zeta^s \nabla^l \eta_s + \\ &+ \frac{1}{\sigma^2} \xi \sigma \cdot \zeta^r \zeta^s \nabla_s \eta_r + \frac{2}{\sigma} \zeta^r \sigma^s \nabla_s \eta_r + \frac{1}{\sigma} (1 + |\zeta|^2) \zeta^b \nabla_b \eta_a \zeta^s \nabla^a \eta_s + \frac{1}{\sigma} \zeta^a \nabla_a \eta_r (\nabla_b \zeta^r + \nabla^r \zeta_b) \zeta^b - \\ &- (\frac{1}{2} - \frac{1}{\sigma} + \frac{1}{\sigma} |\zeta|^2) |\nabla \zeta \xi|^2. \end{aligned}$$

Since

$$(\nabla_a \eta_r \nabla_b \xi^r - \nabla_r \eta_a \nabla^r \eta_b) \zeta^a \zeta^b = -2(\varphi_a^r \nabla_b \eta_r + \varphi_a^r \nabla_r \eta_b) \zeta^a \zeta^b = \frac{1}{\sigma} (\nabla_a \eta_b + \nabla_b \eta_a) \zeta^a \zeta^b,$$

etc., summarizing (4.52), (4.53), (4.55)-(4.57), we get

$$\sigma(\widetilde{scal} - \widetilde{Ric}_{jl} \tilde{\xi}^j \tilde{\xi}^l) = scal - Ric_{jl} \xi^j \xi^l + 4n(\sigma - 1) - \frac{2(n+1)}{\sigma} \Delta_{\mathbb{D}} \sigma - \frac{(n+1)(n-2)}{\sigma^2} |d\sigma|_{\mathbb{D}}^2,$$

from which we obtain (4.49). \square

Corollary 4.5. *For function f on M ,*

$$(4.58) \quad \tilde{\Delta}_{\mathbb{D}} f = \frac{1}{\sigma} \Delta_{\mathbb{D}} f + \frac{n}{\sigma^2} (d\sigma; df)_{\mathbb{D}}$$

Proof. By definition of $\Delta_{\mathbb{D}}$ and $\tilde{\Delta}_{\mathbb{D}}$ we obtain

$$\tilde{\Delta}_{\mathbb{D}} f = (\tilde{g}^{rs} - \tilde{\xi}^r \tilde{\xi}^s) \tilde{\nabla}_r f_s = \frac{1}{\sigma} (g^{rs} - \xi^r \xi^s) (\nabla_r f_s - \tilde{W}_{rs}^a f_a) = \frac{1}{\sigma} \Delta_{\mathbb{D}} f - \frac{1}{\sigma} (g^{rs} - \xi^r \xi^s) \tilde{W}_{rs}^a f_a.$$

Applying (4.51) to the last line, we get (4.58). \square

4.1. \mathbb{D} -homothetic transformations. In this section we consider homothetic gauge transformation, i.e. conformal transformation with constant function. Our main observation here is that these transformations preserve the η -Einstein condition in the paraSasakian case. Moreover, we show that any η -Einstein paraSasakian manifold with $scal \neq 2n$ is \mathbb{D} -homothetic to an Einstein manifold.

Set $\sigma = \alpha = \text{const.}$ in Lemma 4.1 to get

$$(4.59) \quad \bar{g}_{jk} = \alpha g_{jk} + \beta \eta_j \eta_k,$$

where α and $\beta = \alpha(\alpha - 1)$ are constants satisfying $\alpha \neq 0$ and $\alpha + \beta > 0$. The inverse matrix (\bar{g}^{ij}) of (\bar{g}_{jk}) is given by

$$(4.60) \quad \bar{g}^{ij} = \frac{1}{\alpha} g^{ij} - \frac{\beta}{\alpha(\alpha + \beta)} \xi^i \xi^j.$$

Denoting by W_{jk}^i the difference $\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i$ of Christoffel symbols, we have on a paracontact manifold

$$(4.61) \quad W_{jk}^i = -\frac{\beta}{\alpha} (\varphi_j^i \eta_k + \varphi_k^i \eta_j) - \frac{\beta}{2(\alpha + \beta)} \xi^i (\nabla_j \eta_k + \nabla_k \eta_j)$$

which follows from (4.59) and (4.60).

We assume to the end of this subsection that M is a K -paracontact manifold. We have

$$(4.62) \quad W_{jk}^i = -\frac{\beta}{\alpha} (\varphi_j^i \eta_k + \varphi_k^i \eta_j).$$

Substitute (4.62) into $\bar{R}_{ijk}^l = R_{ijk}^l + \nabla_i W_{jk}^l - \nabla_j W_{ik}^l + W_{is}^l W_{jk}^s - W_{js}^l W_{ik}^s$, we obtain

$$(4.63) \quad \begin{aligned} \bar{R}_{ijk}^l &= R_{ijk}^l - \frac{\beta}{\alpha} (2\varphi_k^l \varphi_{ij} - \varphi_j^l \varphi_{ik} + \varphi_i^l \varphi_{jk}) + \\ &+ \frac{\beta}{\alpha} (\nabla_j \varphi_i^l \eta_k + \nabla_j \varphi_k^l \eta_i - \nabla_i \varphi_j^l \eta_k - \nabla_i \varphi_k^l \eta_j) + \frac{\beta^2}{\alpha^2} (\delta_j^l \eta_i \eta_k - \delta_i^l \eta_j \eta_k). \end{aligned}$$

Contracting with respect i and l , we have

$$(4.64) \quad \bar{Ric}_{jk} = Ric_{jk} + 2\frac{\beta}{\alpha} g_{jk} - 2\frac{\beta}{\alpha^2} ((2n+1)\alpha + n\beta) \eta_j \eta_k,$$

where we have used (3.18). Contracting the last equation with (4.60), we get

$$(4.65) \quad \bar{scal} = \frac{1}{\alpha} scal + 2n \frac{\beta}{\alpha^2}.$$

Definition 4.6. If the Ricci tensor of a K -paracontact manifold M is of the form

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

a and b being constant, then M is called an η -Einstein manifold.

Proposition 4.7. Let M be a $(2n+1)$ -dimensional paraSasakian manifold. If the Ricci tensor Ric of M satisfies $Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, then a , b and $scal$ are constant.

Proof. From the assumption on the Ricci tensor Ric we have $a + b = -2n$ and $scal = (2n + 1) + b$. Then we have $Za = -Zb$ and $Z(scal) = (2n + 1)Za + Zb = -2nZb$. On the other hand, *Lemma 3.15* implies $Z(scal) = 2Za + 2(\xi b)\eta(Z) = -2Zb + (\xi b)\eta(Z)$. Therefore, we obtain $(n - 1)(Zb) = -(\xi b)\eta(Z)$. Put $Z = \xi$ in the latter to find $\xi b = 0$. Hence $Zb = 0$, which shows that b is constant. Then a and $scal$ are also constant. \square

Theorem 4.8. *Let $(M, \varphi, \xi, \eta, g)$ be a paracontact manifold and*

$$\bar{\varphi} = \varphi, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\eta} = \alpha\eta, \quad \bar{g} = \alpha g + (\alpha^2 - \alpha)\eta \otimes \eta, \quad \alpha = \text{const.} \neq 0$$

be a \mathbb{D} -homothetic transformation. Then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a paracontact structure too.

- i). If (φ, ξ, η, g) is a K -paracontact structure (resp. paraSasakian), then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also a K -paracontact structure (resp. paraSasakian).*
- ii). If (φ, ξ, η, g) is a η -Einstein paraSasakian structure, then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also a η -Einstein paraSasakian structure.*
- iii). If (φ, ξ, η, g) is a η -Einstein paraSasakian structure with $scal \neq 2n$, then there exists a constant α such that $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an Einstein paraSasakian structure.*

Proof. If ξ is a Killing vector field with respect to g , then $\bar{\xi}$ is also a Killing vector field with respect to \bar{g} , since ξ leaves η invariant. The paraSasakian structure is preserved since the normality conditions is preserved under the \mathbb{D} -homothetic transformations which proves i). If (φ, ξ, η, g) is η -Einstein paraSasakian structure then we have

$$(4.66) \quad Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

It follows from *Proposition 4.7* that a and b are constant and (3.15) yields

$$(4.67) \quad a + b = -2n$$

Then equality (4.66) has the form

$$(4.68) \quad Ric(X, Y) = \left(\frac{scal}{2n} + 1\right)g(X, Y) - \left(2n + 1 + \frac{scal}{2n}\right)\eta(X)\eta(Y),$$

From (4.64) and (4.65), for $\beta = \alpha^2 - \alpha$, we derive

$$(4.69) \quad \bar{Ric}(X, Y) = \left(\frac{\bar{scal}}{2n} + 1\right)\bar{g}(X, Y) - \left(2n + 1 + \frac{\bar{scal}}{2n}\right)\bar{\eta}(X)\bar{\eta}(Y).$$

Finally, we prove iii). If we chose $\alpha = \frac{2n - \overline{scal}}{4n^2 + 4n}$, the equation (4.65) for $\beta = \alpha^2 - \alpha$ gives $\bar{scal} = -2n(2n + 1)$. Then we obtain $\bar{Ric}(X, Y) = -2n\bar{g}(X, Y)$ from (4.69). \square

4.2. Integrable paracontact manifolds. Here we consider the case when the paracomplex structure φ defined on \mathbb{D} is formally integrable, i.e. the Nijenhuis tensor $N_\varphi = [\varphi, \varphi]$ satisfies certain integrability conditions. We see below that in this case the canonical paracontact connection shares many of the properties of the Tanaka-Webster connection on CR -manifold. We begin with

Definition 4.9. An almost paracontact structure (η, φ, ξ) is said to be *integrable* if the almost para-complex structure $\varphi|_{\mathbb{D}}$ satisfies the conditions

$$(4.70) \quad N_\varphi(X, Y) = 0, \quad X, Y \in \Gamma(\mathbb{D}).$$

and

$$(4.71) \quad [\varphi X, Y] + [X, \varphi Y] \in \Gamma(\mathbb{D}), \quad X, Y \in \Gamma(\mathbb{D}).$$

Equivalently, the \pm -eigendistributions \mathbb{D}^\pm of φ are formally integrable in the sense that

$$(4.72) \quad [\mathbb{D}^\pm, \mathbb{D}^\pm] \in \mathbb{D}^\pm.$$

Indeed, (4.72) means that

$$(4.73) \quad \begin{aligned} -\varphi[X + \varphi X, Y + \varphi Y] + [X + \varphi X, Y + \varphi Y] &= 0, \quad X, Y \in \Gamma(\mathbb{D}) \\ \varphi[X - \varphi X, Y - \varphi Y] + [X - \varphi X, Y - \varphi Y] &= 0, \quad X, Y \in \Gamma(\mathbb{D}) \end{aligned}$$

which is clearly equivalent to (4.70)

Theorem 4.10. *A paracontact pseudo-Riemannian manifold $(M, g, \varphi, \eta, \xi)$ is integrable if and only if the canonical paracontact connection preserves the structure tensor φ ,*

$$\tilde{\nabla}\varphi = 0$$

Proof. It suffices to show that the integrability conditions (4.70) and (4.71) are satisfied if and only if

$$(4.74) \quad (\nabla_X \varphi)Y + g(X - hX, Y)\xi - \eta(Y)(X - hX) = 0.$$

Indeed, (4.70) can be also written in the form

$$\varphi[X - \eta(X)\xi, \varphi Y] + \varphi[\varphi X, Y - \eta(Y)\xi] = [\varphi X, \varphi Y] + [X - \eta(X)\xi, Y - \eta(Y)\xi],$$

where $X, Y \in \Gamma(TM)$.

From the last identity, we obtain

$$(4.75) \quad \begin{aligned} &g((\nabla_{\varphi X} \varphi)Y, Z) - g((\nabla_{\varphi Y} \varphi)X, Z) + g((\nabla_X \varphi)Y, \varphi Z) - g((\nabla_Y \varphi)X, \varphi Z) - \\ &- 2d\eta(X, Y)\xi + \eta(X)(\nabla_Y Z - \eta(Y)(\nabla_X Z) + \eta(X)g(\nabla_{\varphi Y} \xi, \varphi Z) - \eta(Y)g(\nabla_{\varphi X} \xi, \varphi Z) - \\ &- \eta(X)(g(\nabla_\xi \varphi Y, \varphi Z) + g(\nabla_\xi Y, Z)) + \eta(Y)(g(\nabla_\xi \varphi X, \varphi Z) + g(\nabla_\xi X, Z)) = 0 \end{aligned}$$

From the identity $\nabla_\xi \varphi = 0$ and the Lemma 2.5, we get

$$\begin{aligned} &g((\nabla_{\varphi X} \varphi)Y, Z) - g((\nabla_{\varphi Y} \varphi)X, Z) + g((\nabla_X \varphi)Y, \varphi Z) - g((\nabla_Y \varphi)X, \varphi Z) + \\ &+ \eta(X)((\nabla_Y Z) + (\nabla_Z Y) - \eta(Y)((\nabla_X Z) + (\nabla_Z X) - 2d\eta(X, Y)\xi) = 0 \end{aligned}$$

That is,

$$(4.76) \quad \begin{aligned} &\varphi_k^h \nabla_i \varphi_{hj} - \varphi_k^h \nabla_j \varphi_{hi} + \varphi_i^s \nabla_s \varphi_{kj} - \varphi_j^s \nabla_s \varphi_{si} + \eta_i(\nabla_j \eta_k + \nabla_k \eta_j) - \\ &- \eta_j(\nabla_i \eta_k + \nabla_k \eta_i) - 2\varphi_{ij} \eta_k = 0. \end{aligned}$$

Since $d\eta$ is closed, the third and the fourth terms of the left-hand side of (4.76) are calculate as follows:

$$\begin{aligned} &\varphi_i^s \nabla_s \varphi_{kj} - \varphi_j^s \nabla_s \varphi_{si} = -\varphi_i^s (\nabla_k \varphi_{js} + \nabla_j \varphi_{sk}) + \varphi_j^s (\nabla_k \varphi_{is} + \nabla_i \varphi_{sk}) = \nabla_i(\eta_j \eta_k) - \\ &- \nabla_j(\eta_i \eta_k) + \nabla_k(\eta_i \eta_j) - 2\varphi_j^s \nabla_k \varphi_{si} + \varphi_k^s \nabla_j \varphi_{si} - \varphi_k^s \nabla_i \varphi_{sj}. \end{aligned}$$

Therefore (4.76) is equivalent to

$$(4.77) \quad \varphi_j^s \nabla_k \varphi_{si} - \eta_i \nabla_k \eta_j = 0.$$

From the identity (4.77) it follows that

$$\begin{aligned}\tilde{\nabla}_i \varphi_{kj} &= \nabla_i \varphi_{kj} + \nabla_i \eta_s \varphi_j^s \eta_k - \nabla_i \eta_s \varphi_k^s \eta_j = \nabla_i \varphi_{kj} + \varphi_j^s \varphi_s^r \nabla_i \varphi_{rk} - \varphi_k^s \nabla_i \eta_s \eta_j \\ &= \nabla_i \varphi_{kj} + \nabla_i \varphi_{jk} - \eta_j (\xi^r \nabla_i \varphi_{rk} + \varphi_k^s \nabla_i \eta_s) = -\eta_j \nabla_i (\eta_s \varphi_k^s) = 0.\end{aligned}$$

□

The obstruction an integrable paracontact manifold to be normal, i.e. paraSasakian, is encoded in the (horizontal) torsion $T(\xi, X) = h(X)$, $X \in \mathbb{D}$ of the canonical paracontact connection. The main result here is the following

Theorem 4.11. *The torsion of the canonical paracontact connection vanishes on an integrable paracontact manifold if and only if it is paraSasakian.*

Proof. Let be (M, g, η) an integrable paracontact manifold. From the *Theorem 4.10* follows $\tilde{\nabla}\varphi = 0$. Since equation (4.45) we get

$$(4.78) \quad T_{ijk} = -\eta_i \varphi_{jk} + \eta_j \varphi_{ik} - \eta_j \nabla_i \eta_k + \eta_i \nabla_j \eta_k + 2\varphi_{ij} \eta_k.$$

From the last equation we calculate

$$(4.79) \quad \xi^i T_{ijk} = -\varphi_{jk} + \nabla_j \eta_k = \frac{1}{2}(\nabla_j \eta_k + \nabla_k \eta_j).$$

If M is a paraSasakian manifold, then $P_{rsi} = \nabla_r \varphi_{si} - \eta_i g_{rs} + \eta_s g_{ri} = 0$. From the last equation we calculate

$$(4.80) \quad \varphi_s^i \nabla_r \eta_i = g_{rs} - \eta_r \eta_s.$$

Transvecting (4.80) by φ_k^s , we obtain

$$(4.81) \quad \nabla_r \eta_k = \varphi_{rk}.$$

From the equation (4.81) we get

$$\xi^i T_{ijk} = \frac{1}{2}(\nabla_j \eta_k + \nabla_k \eta_j) = \frac{1}{2}(\varphi_{rk} + \varphi_{kr}) = 0.$$

If $\xi^i T_{ijk} = 0$ from equation (4.79) we have

$$(4.82) \quad \nabla_j \eta_k = \varphi_{jk}.$$

From equations (4.77) and (4.82) we calculate

$$\begin{aligned}P_{rsi} &= \nabla_r \varphi_{si} - \eta_i g_{rs} + \eta_s g_{ri} = \xi^l \eta_s \nabla_r \varphi_{li} + \eta_i \varphi_s^k \nabla_r \eta_k - \eta_i g_{rs} + \eta_s g_{ri} = -\xi^l \varphi_i^l \nabla_r \eta_l + \\ &+ \eta_i \varphi_s^k \nabla_r \eta_k - \eta_i g_{rs} + \eta_s g_{ri} = -\eta_s \varphi_{rl} \varphi_i^l + \eta_i \varphi_{rk} \varphi_s^k - \eta_i g_{rs} + \eta_s g_{ri} = \eta_s (-g_{ri} + \eta_r \eta_i) - \\ &- \eta_i (-g_{rs} + \eta_r \eta_s) - \eta_i g_{rs} + \eta_s g_{ri} = \eta_i g_{rs} - \eta_s g_{ri} - \eta_i g_{rs} + \eta_s g_{ri} = 0.\end{aligned}$$

Therefore $P = 0$, which equivalent to M to be a paraSasakian manifold. □

5. PARACONTACT MANIFOLDS WITH TORSION

If we introduce the forms

$$(5.83) \quad dF^-(X, Y, Z) = dF(\varphi X, Y, Z) + dF(X, \varphi Y, \varphi Z) + dF(\varphi X, Y, \varphi Z) + dF(X, Y, Z);$$

$$(5.84) \quad dF^\varphi(X, Y, Z) = -dF(\varphi X, \varphi Y, \varphi Z).$$

and a direct consequence of the definitions and *Proposition 2.4* is the following

Proposition 5.1. *On any almost paracontact manifold the identities hold:*

$$(5.85) \quad dF^-(X, Y, Z) = -N^{(1)}(X, Y, \varphi Z) - N^{(1)}(Y, Z, \varphi X) - N^{(1)}(Z, X, \varphi Y);$$

$$(5.86) \quad N^{(1)}(X, Y, Z) = N^{(1)}(\varphi X, \varphi Y, Z) + \eta(Y)N^{(1)}(X, \xi, Z) + \eta(X)N^{(1)}(\xi, Y, Z);$$

$$(5.87) \quad N^{(1)}(X, Y) = (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X + (\nabla_X\varphi)\varphi Y - (\nabla_Y\varphi)\varphi X - \eta(X)\nabla_Y\xi + \eta(Y)\nabla_X\xi.$$

Definition 5.2. A linear connection $\bar{\nabla}$ is said to be an almost paracontact connection if it preserves the almost paracontact structure:

$$\bar{\nabla}g = \bar{\nabla}\eta = \bar{\nabla}\varphi = 0.$$

Theorem 5.3. *Let $(M^{(2n+1)}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. The following conditions are equivalent:*

- 1) *The tensor $N^{(1)}$ is skew-symmetric and ξ is a Killing vector field.*
- 2) *There exists an almost paracontact linear connection $\bar{\nabla}$ with totally skew-symmetric torsion tensor T .*

Moreover, this connection is unique and determined by

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}T(X, Y, Z),$$

where the torsion T is defined by

$$T = 2\eta \wedge d\eta + d^\varphi F - N^{(1)} + \eta \wedge (\xi \lrcorner N^{(1)}).$$

Proof. Let assume that such a connection exists. Then

$$0 = g(\nabla_X \xi, Z) + \frac{1}{2}T(X, \xi, Z)$$

holds and the skew-symmetric of T yields that ξ is a Killing vector field, $2d\eta = \xi \lrcorner T$, $\xi \lrcorner d\eta = 0$ and

$$T(\varphi X, \varphi Y, Z) + T(\varphi X, Y, \varphi Z) + T(X, \varphi Y, \varphi Z) + T(X, Y, Z) = -N^{(1)}(X, Y, Z).$$

The latter formula shows that $N^{(1)}$ is skew-symmetric. Since φ is $\bar{\nabla}$ -parallel, we can express the Riemannian covariant derivative of φ by the torsion form:

$$T(X, \varphi Y, Z) + T(X, Y, \varphi Z) = -2g((\nabla_X \varphi)Y, Z).$$

Taking the cyclic sum in the above equality, we obtain

$$\sigma_{X,Y,Z}T(X, Y, \varphi Z) = -\sigma_{X,Y,Z}g((\nabla_X \varphi)Y, Z) = dF(X, Y, Z).$$

Adding this result to the formula expressing the tensor $N^{(1)}$ by the torsion T , come calculations yield

$$T(\varphi X, \varphi Y, \varphi Z) = -dF(X, Y, Z) - g(N^{(1)}(X, Y), \varphi Z) + \eta(Z)N^{(2)}(X, Y).$$

By replacing X, Y, Z by $\varphi X, \varphi Y, \varphi Z$ and using the symmetry property of the tensor $N^{(1)}$ in *Proposition 5.1*, we obtain the formula for the torsion tensor T .

For the converse, suppose that the almost paracontact structure has properties 1) and define the connection $\bar{\nabla}$ by the formulas 2). Clearly T is skew-symmetric and $2d\eta = \xi \lrcorner T = 2\nabla\eta$. Since ξ is a Killing vector field, we conclude $\nabla g = \nabla\xi = 0$. Furthermore, using the conditions 1) and *Proposition 5.1*, we obtain $\xi \lrcorner dF = N^{(2)}$. Finally we have to prove that $\nabla\varphi = 0$. This follows by straightforward computations using the relation between $\nabla\varphi$ and the torsion tensor T , *Proposition 5.1*, as well as the following lemma. \square

Lemma 5.4. *Let $(M^{(2n+1)}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold with a totally skew-symmetric tensor $N^{(1)}$. Then the following equalities hold:*

$$(5.88) \quad \nabla_\xi \xi = \xi \lrcorner d\eta = 0;$$

$$(5.89) \quad (\nabla_X \eta)Y + (\nabla_Y \eta)X = -(\nabla_{\varphi X} \eta)\varphi Y - (\nabla_{\varphi Y} \eta)\varphi X;$$

$$(5.90) \quad \begin{aligned} N^{(1)}(\varphi X, Y, \xi) &= N^{(1)}(X, \varphi Y, \xi) = -N^{(2)}(X, Y) = \\ &= dF(X, Y, \xi) = dF(\varphi X, \varphi Y, \xi). \end{aligned}$$

Proof. The identities follow from *Proposition 2.4*, *Lemma 2.7* and formula (5.86). \square

We discuss these results for some special paracontact structures.

Theorem 5.5. *Let $(M^{(2n+1)}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold with totally skew-symmetric tensor $N^{(1)} = 0$. Then the condition $dF = 0$ implies $N^{(1)} = 0$.*

1) *A paracontact metric structure $(F = d\eta)$ admits an almost paracontact connection with totally skew-symmetric torsion if and only if it is paraSasakian. In this case, the connection is unique, its torsion is given by*

$$T = 2\eta \wedge d\eta$$

and T is parallel, $\bar{\nabla}T = 0$.

2) *A normal ($N^{(1)} = 0$) paracontact metric structure admits a unique almost paracontact connection with totally skew-symmetric torsion if and only if ξ is Killing vector field. The torsion T is given by*

$$T = 2\eta \wedge d\eta + d^\varphi F.$$

Proof. If $dF = 0$, *Lemma 5.4* implies that $N^{(2)} = \xi \lrcorner N^{(1)} = 0$. Then *Proposition* leads to $0 = dF^-(X, Y, Z) = -3N^{(1)}(\varphi X, Y, Z)$. The assertion that $\bar{\nabla}T = 0$ in a paraSasakian manifold follows by direct verification. \square

We introduce the forms

$$(5.91) \quad \rho^{\bar{\nabla}}(X, Y) = \frac{1}{2} \sum_{i=1}^{2n+1} R^{\bar{\nabla}}(X, Y, e_i, \varphi e_i);$$

$$(5.92) \quad t(X) = \frac{1}{2} \sum_{i=1}^{2n+1} T(X, e_i, \varphi e_i);$$

$$(5.93) \quad dt(X, Y) = \frac{1}{2} \sum_{i=1}^{2n+1} dT(X, Y, e_i, \varphi e_i);$$

Proposition 5.6. *Let $(M^{(2n+1)}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold with totally skew-symmetric tensor $N^{(1)}$ and Killing vector ξ . Let $\bar{\nabla}$ be the unique almost paracontact connection with totally skew-symmetric torsion. Then one has*

$$(5.94) \quad \rho^{\bar{\nabla}}(X, Y) = Ric^{\bar{\nabla}}(X, \varphi Y) + (\bar{\nabla}_X t)Y + \frac{1}{2}dt(X, Y)$$

Proof. We follow the scheme in [3] and use the curvature properties of $\bar{\nabla}$ in to calculate $dt(X, Y)$:

$$dt(X, Y) = (\bar{\nabla}_X t)Y - (\bar{\nabla}_Y t)X + \sigma^T(X, Y, e_i, \varphi e_i) - (\bar{\nabla}_{\varphi e_i} T)(X, Y, e_i).$$

The first Bianchi identity for $\bar{\nabla}$ together with the latter identity implies

$$4\rho^{\bar{\nabla}}(X, Y) + 2Ric^{\bar{\nabla}}(Y, \varphi X) - 2Ric^{\bar{\nabla}}(X, \varphi Y) = 2dt(X, Y) + 2(\bar{\nabla}_X t)Y - 2(\bar{\nabla}_Y t)X.$$

Using the relation between the curvature tensors of ∇ and $\bar{\nabla}$, we obtain

$$Ric^{\bar{\nabla}}(Y, \varphi X) + Ric^{\bar{\nabla}}(X, \varphi Y) = -(\bar{\nabla}_X t)Y - (\bar{\nabla}_Y t)X.$$

The last two equalities lead to the desired formula. \square

Proposition 5.7. *Let $(M^{(2n+1)}, \varphi, \xi, \eta, g)$ be a paraSasakian metric manifold and ∇ be the unique almost paracontact connection with totally skew-symmetric torsion. Then one has*

$$(5.95) \quad \rho^{\bar{\nabla}}(X, \varphi Y) = Ric^{\bar{\nabla}}(X, Y) + 4(n-1)(g(X, Y) - \eta(X)\eta(Y))$$

Moreover, the 2-form $\rho^{\bar{\nabla}} = 0$ if and only if

$$(5.96) \quad Ric(X, Y) = -2(2n-1)g(X, Y) + 2(n-1)\eta(X)\eta(Y).$$

Proof. On a paraSasakian manifold $T = 2\eta \wedge d\eta = 2\eta \wedge F$ and $\nabla T = 0$, where $F(X, Y) = g(X, \varphi Y)$ is the fundamental form of the paraSasakian structure. Consequently, we calculate that

$$\bar{\nabla} t = 0, dt = 8(n-1)F, \sum_{i=1}^{2n+1} g(T(X, e_i), T(Y, e_i)) = -8g(X, Y) - 8(n-1)\eta(X)\eta(Y).$$

Using the relation between the curvature tensors of ∇ and $\bar{\nabla}$, we obtain

$$Ric(X, Y) = Ric^{\bar{\nabla}}(X, Y) - 2g(X, Y) - 2(n-1)\eta(X)\eta(Y)$$

and

$$\rho(X, \varphi Y) = \rho^{\overline{\nabla}}(X, \varphi Y) - (2n - 1)(g(X, Y) - \eta(X)\eta(Y)),$$

and the proof follows from *Proposition 5.6*. \square

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